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BIHERMITIAN SUPERSYMMETRIC QUANTUM MECHANICS

by

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Abstract

BiHermitian geometry, discovered long ago by Gates, Hull and Roček, is the most general sigma model target space geometry allowing for $(2, 2)$ world sheet supersymmetry. In this paper, we work out supersymmetric quantum mechanics for a biHermitian target space. We display the full supersymmetry of the model and illustrate in detail its quantization procedure. Finally, we show that the quantized model reproduces the Hodge theory for compact twisted generalized Kaehler manifolds recently developed by Gualtieri in [33]. This allows us to recover and put in a broader context the results on the biHermitian topological sigma models obtained by Kapustin and Li in [9].

Keywords: quantum field theory in curved spacetime; geometry, differential geometry and topology.

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1 Introduction

In a classic paper, Gates, Hull and Roček [1] showed that, for a 2–dimensional sigma model, the most general target space geometry allowing for $(2,2)$ supersymmetry was biHermitian or Kaehler with torsion geometry. This is characterized by a Riemannian metric g_{ab}, two generally non commuting complex structures $K_\pm{}^a{}_b$ and a closed 3–form H_{abc}, such that g_{ab} is Hermitian with respect to both the $K_\pm{}^a{}_b$ and the $K_\pm{}^a{}_b$ are parallel with respect to two different metric connections with torsion proportional to $\pm H_{abc}$ [2–5].

$(2,2)$ superconformal sigma models with Calabi–Yau target manifolds describe of compactifications of type II superstring and are therefore of considerable physical interest. These are however nonlinear interacting field theories and, so, are rather complicated and difficult to study. In 1988, Witten showed that a $(2,2)$ supersymmetric sigma model on a Calabi–Yau space could be twisted in two different ways, to yield the so called A and B topological sigma models [6,7]. Unlike the original untwisted sigma model, the topological models are soluble field theories: the calculation of observables can be carried out by standard methods of geometry and topology. For the A model, the ring of observables is found to be a deformation of the complex de Rham cohomology $\bigoplus_p H^p(M,\mathbb{C})_{\mathrm{qu}}$, going under the name of quantum cohomology, and all correlators can be shown to be symplectic invariants of M. For the B model, the ring of observables turns out to be isomorphic to $\bigoplus_{p,q} H^p(\wedge^q T^{1,0}M)$ and all correlators are invariants of the complex structure on M. Topological twisting of Calabi–Yau $(2,2)$ supersymmetric sigma models is therfore a very useful field theoretic procedure for the study of such field theories.

Witten's analysis was restricted to the case where the sigma model target space geometry was Kaehler. This geometry is less general than that considered by Gates, Hull and Roček, as it corresponds to the case where $K_+{}^a{}_b = \pm K_-{}^a{}_b$ and $H_{abc} = 0$. In the last few years, many attempts have been made to construct sigma models with a biHermitian target manifolds, by invoking world sheet supersymmetry, employing the Batalin–Vilkovisky quantization algorithm, etc. [8–24]. A turning point in the quest towards accomplishing this goal was the realization that biHermitian geometry is naturally expressed in the language of generalized complex and Kaehler geometry worked out by Hitchin and Gualtieri [25–28].

The subject of topological field theory itself can be traced back to Witten's fundamental work on dynamical supersymmetry breaking [29,30]. Those findings led naturally to a reformulation of de Rham and Morse theory as supersymmetric quantum mechanics [31]. Since then, supersymmetric quantum mechanics has been the object of intense study, for the rich relation existing between the amount of 1–dimensional supersymmetry and the type of the differential geometric structure (Riemannian, Kaehler, hyperKaehler, etc.) present in target space (see e. g. [32] and references therein).

In this paper, we analyze supersymmetric quantum mechanics with a biHermitian target space. This model was first considered by Kapustin and Li in [9] and was used to study the topological sector of $N = 2$ sigma model with H flux. Here, we continue its study covering aspects not touched by Kapustin's and Li's analysis. We display the full supersymmetry of biHermitian supersymmetric quantum mechanics. Further, we illustrate in detail the quantization procedure and show that demanding that the supersymmetry algebra is satisfied at the quantum level solves all quantum ordering ambiguities. Finally, we show that, upon quantization, the model reproduces the Hodge theory for compact twisted generalized Kaehler manifolds recently developed by Gualtieri [33] (see also [34]), thereby generalizing the well–known correspondence holding for ordinary Kaehler geometry. We obtain in this way explicit local coordinate expressions of the relevant differential operators of Gualtieri's theory, which may be useful in applications. We also explore the implications of our findings for the geometrical interpretation of the biHermitian topological sigma models [21, 22]. In this way, we recover and put in a broader context the results obtained by Kapustin and Li in [9].

2 BiHermitian geometry

The target space geometry of the supersymmetric quantum mechanics studied in the paper is biHermitian. Below, we review the basic facts of biHermitian geometry.

Let M be a smooth manifold. A biHermitian structure $(g, H, K_\pm)$ on M consists of the following elements.

a) A Riemannian metric g_{ab} [1].

b) A closed 3–form H_{abc}

$$\partial_{[a} H_{bcd]} = 0. \tag{2.1}$$

c) Two complex structures $K_\pm{}^a{}_b$,

$$K_\pm{}^a{}_c K_\pm{}^c{}_b = -\delta^a{}_b, \tag{2.2}$$

$$K_\pm{}^d{}_a \partial_d K_\pm{}^c{}_b - K_\pm{}^d{}_b \partial_d K_\pm{}^c{}_a - K_\pm{}^c{}_d \partial_a K_\pm{}^d{}_b + K_\pm{}^c{}_d \partial_b K_\pm{}^d{}_a = 0. \tag{2.3}$$

They satisfy the following conditions.

d) g_{ab} is Hermitian with respect to both $K_\pm{}^a{}_b$:

$$K_{\pm ab} + K_{\pm ba} = 0. \tag{2.4}$$

e) The complex structures $K_\pm{}^a{}_b$ are parallel with respect to the connections $\nabla_{\pm a}$

$$\nabla_{\pm a} K_\pm{}^b{}_c = 0, \tag{2.5}$$

where the connection coefficients $\Gamma_\pm{}^a{}_{bc}$ are given by

$$\Gamma_\pm{}^a{}_{bc} = \Gamma^a{}_{bc} \pm \frac{1}{2} H^a{}_{bc}, \tag{2.6}$$

$\Gamma^a{}_{bc}$ being the usual Levi–Civita connection coefficients.

The connections $\nabla_{\pm a}$ have a non vanishing torsion $T_{\pm abc}$, which is totally antisymmetric and in fact equal to the 3–form H_{abc} up to sign,

$$T_{\pm abc} = \pm H_{abc}. \tag{2.7}$$

[1] Here and below, indices are raised and lowered by using the metric g_{ab}.

The Riemann tensors $R_{\pm abcd}$ of the $\nabla_{\pm a}$ satisfy a number of relations, the most relevant of which are collected in appendix A.

In biHermitian geometry, one is dealing with two generally non commuting complex structures. As it turns out, it is not convenient to write the relevant tensor identities in the complex coordinates associated with either of them. General coordinates are definitely more natural and yield a more transparent formalism. Having this in mind, we define the complex tensors

$$\Lambda_\pm{}^a{}_b = \frac{1}{2}\big(\delta^a{}_b - iK_\pm{}^a{}_b\big). \tag{2.8}$$

The $\Lambda_\pm{}^a{}_b$ satisfy the relations

$$\Lambda_\pm{}^a{}_c\Lambda_\pm{}^c{}_b = \Lambda_\pm{}^a{}_b, \tag{2.9a}$$

$$\Lambda_\pm{}^a{}_b + \overline{\Lambda}_\pm{}^a{}_b = \delta^a{}_b, \tag{2.9b}$$

$$\Lambda_\pm{}^a{}_b = \overline{\Lambda}_{\pm b}{}^a. \tag{2.9c}$$

Thus, the $\Lambda_\pm{}^a{}_b$ are projector valued endomorphisms of the complexified tangent bundle $TM \otimes \mathbb{C}$. The corresponding projection subbundles of $TM \otimes \mathbb{C}$ are the $K_\pm$–holomorphic tangent bundles $T_\pm^{1,0}M$.

It turns out that the 3–form H_{abc} is of type $(2,1)+(1,2)$ with respect to both complex structures $K_\pm{}^a{}_b$,

$$H_{def}\Lambda_\pm{}^d{}_a\Lambda_\pm{}^e{}_b\Lambda_\pm{}^f{}_c = 0 \quad \text{and c.c.} \tag{2.10}$$

Other relations of the same type involving the Riemann tensors $R_{\pm abcd}$ are collected in appendix A.

In [26], Gualtieri has shown that biHermitian geometry is related to generalized Kaehler geometry. This, in turn, is part of generalized complex geometry. For a review of generalized complex and Kaehler geometry accessible to physicists, see [27, 28].

3 The (2,2) supersymmetric sigma model

We shall review next the main properties of the biHermitian (2,2) supersymmetric sigma model, which are relevant in the following analysis.

The base space of the model is a $1+1$ dimensional Minkoskian surface Σ, usually taken to be a cylinder. The target space of the model is a smooth manifold M equipped with a biHermtian structure $(g, H, K_\pm)$. The basic fields of the model are the embedding field x^a of Σ into M and the spinor field ${\psi_\pm}^a$, which is valued in the vector bundle x^*TM [2].

The action of biHermitian (2,2) supersymmetric sigma model is given by

$$\begin{aligned} S = \int_\Sigma d^2\sigma \bigg[&\frac{1}{2}(g_{ab}+b_{ab})(x)\partial_{++}x^a\partial_{--}x^b \\ &+ \frac{i}{2}g_{ab}(x)({\psi_+}^a\nabla_{+\,--}{\psi_+}^b + {\psi_-}^a\nabla_{-\,++}{\psi_-}^b) \\ &+ \frac{1}{4}R_{+abcd}(x){\psi_+}^a{\psi_+}^b{\psi_-}^c{\psi_-}^d\bigg], \end{aligned} \tag{3.1}$$

where $\partial_{\pm\pm} = \partial_0 \pm \partial_1$,

$$\nabla_{\pm\,\mp\mp} = \partial_{\mp\mp} + {\Gamma_\pm}^{\cdot}{}_{c\,\cdot}(x)\partial_{\mp\mp}x^c \tag{3.2}$$

and the field b_{ab} is related to H_{abc} as

$$H_{abc} = \partial_a b_{bc} + \partial_b b_{ca} + \partial_c b_{ab}. \tag{3.3}$$

The $(2,2)$ supersymmetry variations of the basic fields can be written in several ways. We shall write them in the following convenient form

$$\begin{aligned} \delta_S x^a = i\big[&\alpha^+{\Lambda_+}^a{}_b(x){\psi_+}^b + \tilde{\alpha}^+{\overline{\Lambda}_+}{}^a{}_b(x){\psi_+}^b \\ &+ \alpha^-{\Lambda_-}^a{}_b(x){\psi_-}^b + \tilde{\alpha}^-{\overline{\Lambda}_-}{}^a{}_b(x){\psi_-}^b\big], \end{aligned} \tag{3.4a}$$

[2] Complying with an established use, here and in the following the indices $\pm$ are employed both to label the two complex structures $K_\pm$ of the relevant biHermitian structure and to denote 2–dimensional spinor indices. It should be clear from the context what they stand for and no confusion should arise.

$$\delta_S\psi_\pm{}^a = -\alpha^\pm\overline{\Lambda}_\pm{}^a{}_b(x)\partial_{\pm\pm}x^b - \tilde{\alpha}^\pm\Lambda_\pm{}^a{}_b(x)\partial_{\pm\pm}x^b \tag{3.4b}$$
$$- i\Gamma_\pm{}^a{}_{bc}(x)\big[\alpha^+\Lambda_+{}^b{}_d(x)\psi_+{}^d + \tilde{\alpha}^+\overline{\Lambda}_+{}^b{}_d(x)\psi_+{}^d$$
$$+ \alpha^-\Lambda_-{}^b{}_d(x)\psi_-{}^d + \tilde{\alpha}^-\overline{\Lambda}_-{}^b{}_d(x)\psi_-{}^d\big]\psi_\pm{}^c$$
$$\pm iH^a{}_{bc}(x)\big[\alpha^\pm\Lambda_\pm{}^b{}_d(x)\psi_\pm{}^d + \tilde{\alpha}^\pm\overline{\Lambda}_\pm{}^b{}_d(x)\psi_\pm{}^d\big]\psi_\pm{}^c$$
$$\mp \frac{i}{2}\big(\alpha^\pm\Lambda_\pm{}^a{}_d + \tilde{\alpha}^\pm\overline{\Lambda}_\pm{}^a{}_d\big)H^d{}_{bc}(x)\psi_\pm{}^b\psi_\pm{}^c,$$

where $\alpha^\pm$, $\tilde{\alpha}^\pm$ are constant Grassmann parameters. δ_S generates a $(2,2)$ supersymmetry algebra on shell. The action S enjoys $(2,2)$ supersymmetry, so that

$$\delta_S S = 0. \tag{3.5}$$

The biHermitian $(2,2)$ supersymmetric sigma model is characterized also by two types of R symmetry: the $U(1)_V$ vector and the $U(1)_A$ axial R symmetries

$$\delta_R x^a = 0, \tag{3.6a}$$
$$\delta_R\psi_\pm{}^a = i(\epsilon_V \pm \epsilon_A)\Lambda_\pm{}^a{}_b(x)\psi_\pm{}^b - i(\epsilon_V \pm \epsilon_A)\overline{\Lambda}_\pm{}^a{}_b(x)\psi_\pm{}^b, \tag{3.6b}$$

where ϵ_V, ϵ_A are infinitesimal real parameters. Classically, the action S enjoys both types of R symmetry, so that

$$\delta_R S = 0. \tag{3.7}$$

As is well known, at the quantum level, the R symmetries are spoiled by anomalies unless certain topological conditions on the target manifold M are satisfied [9].

It is convenient to introduce the projected spinor fields

$$\chi_\pm{}^a = \Lambda_\pm{}^a{}_b(x)\psi_\pm{}^b, \tag{3.8a}$$
$$\overline{\chi}_\pm{}^a = \overline{\Lambda}_\pm{}^a{}_b(x)\psi_\pm{}^b. \tag{3.8b}$$

In terms of these, the action S reads

$$S = \int_\Sigma d^2\sigma\bigg[\frac{1}{2}(g_{ab} + b_{ab})(x)\partial_{++}x^a\partial_{--}x^b \tag{3.9}$$
$$+ ig_{ab}(x)(\overline{\chi}_+{}^a\nabla_{+--}\chi_+{}^b + \overline{\chi}_-{}^a\nabla_{-++}\chi_-{}^b)$$
$$+ R_{+abcd}(x)\overline{\chi}_+{}^a\chi_+{}^b\overline{\chi}_-{}^c\chi_-{}^d\bigg].$$

The $(2,2)$ supersymmetry variations (3.4) take the simpler form

$$\delta_S x^a = i\big[\alpha^+ \chi_+{}^a + \tilde{\alpha}^+ \overline{\chi}_+{}^a + \alpha^- \chi_-{}^a + \tilde{\alpha}^- \overline{\chi}_-{}^a\big], \tag{3.10a}$$

$$\begin{aligned} \delta_S \chi_\pm{}^a = {} & -i\Gamma_\pm{}^a{}_{bc}(x)\big[\alpha^+ \chi_+{}^b + \tilde{\alpha}^+ \overline{\chi}_+{}^b + \alpha^- \chi_-{}^b + \tilde{\alpha}^- \overline{\chi}_-{}^b\big]\chi_\pm{}^c \\ & \pm \frac{i}{2}\alpha^\pm H^a{}_{bc}(x)\chi_\pm{}^b\chi_\pm{}^c - \tilde{\alpha}^\pm \Lambda_\pm{}^a{}_b(x)\big[\partial_{\pm\pm}x^b \mp iH^b{}_{cd}(x)\overline{\chi}_\pm{}^c\chi_\pm{}^d\big], \end{aligned} \tag{3.10b}$$

$$\begin{aligned} \delta_S \overline{\chi}_\pm{}^a = {} & -i\Gamma_\pm{}^a{}_{bc}(x)\big[\alpha^+ \chi_+{}^b + \tilde{\alpha}^+ \overline{\chi}_+{}^b + \alpha^- \chi_-{}^b + \tilde{\alpha}^- \overline{\chi}_-{}^b\big]\overline{\chi}_\pm{}^c \\ & \pm \frac{i}{2}\tilde{\alpha}^\pm H^a{}_{bc}(x)\overline{\chi}_\pm{}^b\overline{\chi}_\pm{}^c - \alpha^\pm \overline{\Lambda}_\pm{}^a{}_b(x)\big[\partial_{\pm\pm}x^b \mp iH^b{}_{cd}(x)\chi_\pm{}^c\overline{\chi}_\pm{}^d\big]. \end{aligned} \tag{3.10c}$$

Similarly, the R symmetry (3.6) can be cast in simple form as

$$\delta_R x^a = 0, \tag{3.11a}$$

$$\delta_R \chi_\pm{}^a = +i(\epsilon_V \pm \epsilon_A)\chi_\pm{}^a, \tag{3.11b}$$

$$\delta_R \overline{\chi}_\pm{}^a = -i(\epsilon_V \pm \epsilon_A)\overline{\chi}_\pm{}^a. \tag{3.11c}$$

This projected spinor formulation of the $(2,2)$ supersymmetric sigma model turns out to be far more convenient in the following analysis than the more conventional one reviewed in the first half of this section.

4 The biHermitian susy quantum mechanics

We can obtain the biHermitian supersymmetric quantum mechanics from the biHermitian $(2,2)$ supersymmetric sigma model by taking the world sheet Σ to be of the form $\Sigma = T \times S^1$ with $T = \mathbb{R}$ and dimensionally reduce form $1+1$ to $1+0$ by shrinking the S^1 factor to a point.

We use the projected spinor formalism illustrated in section 3. Then, the biHermitian supersymmetric quantum mechanics action S_{QM} reads

$$S_{QM} = \int_T dt \bigg[\frac{1}{2} g_{ab}(x)\partial_t x^a \partial_t x^b + i g_{ab}(x)(\overline{\chi}_+{}^a \nabla_{+t}\chi_+{}^b + \overline{\chi}_-{}^a \nabla_{-t}\chi_-{}^b) \\ + R_{+abcd}(x)\overline{\chi}_+{}^a \chi_+{}^b \overline{\chi}_-{}^c \chi_-{}^d\bigg], \tag{4.1}$$

where the nabla operator $\nabla_{\pm t}$ is given by

$$\nabla_{\pm t} = \partial_t + \Gamma_{\pm}{}^{\cdot}{}_{c\cdot}(x)\partial_t x^c. \tag{4.2}$$

The b field no longer appears in the action, as is obvious from dimensional considerations. Note that, by (3.8), the fermionic variables $\chi_\pm{}^a$, $\overline{\chi}_\pm{}^a$ are constrained: $\overline{\Lambda}_\pm{}^a{}_b(x)\chi_\pm{}^b = \Lambda_\pm{}^a{}_b(x)\overline{\chi}_\pm{}^b = 0$.

The supersymmetry variations are easily read off from (3.10). It is convenient to decompose supersymmetry variation operator δ_S as

$$\delta_S = \alpha^+ q_+ + \tilde{\alpha}^+ \overline{q}_+ + \alpha^- q_- + \tilde{\alpha}^- \overline{q}_-, \tag{4.3}$$

where the fermionic variation operators $q_\pm$, $\overline{q}_\pm$ are given by

$$q_\pm x^a = i\chi_\pm{}^a, \tag{4.4a}$$

$$\overline{q}_\pm x^a = i\overline{\chi}_\pm{}^a, \tag{4.4b}$$

$$q_\pm \chi_\pm{}^a = 0, \tag{4.4c}$$

$$q_\mp \chi_\pm{}^a = -i\Gamma_\pm{}^a{}_{bc}(x)\chi_\mp{}^b \chi_\pm{}^c, \tag{4.4d}$$

$$\overline{q}_\pm \chi_\pm{}^a = -i\Gamma_\pm{}^a{}_{bc}(x)\overline{\chi}_\pm{}^b \chi_\pm{}^c - \Lambda_\pm{}^a{}_b(x)\big[\partial_t x^b \mp iH^b{}_{cd}(x)\overline{\chi}_\pm{}^c \chi_\pm{}^d\big], \tag{4.4e}$$

$$\overline{q}_\mp \chi_\pm{}^a = -i\Gamma_\pm{}^a{}_{bc}(x)\overline{\chi}_\mp{}^b \chi_\pm{}^c, \tag{4.4f}$$

$$q_\pm \overline{\chi}_\pm{}^a = -i\Gamma_\pm{}^a{}_{bc}(x)\chi_\pm{}^b\overline{\chi}_\pm{}^c - \overline{\Lambda}_\pm{}^a{}_b(x)\big[\partial_t x^b \mp iH^b{}_{cd}(x)\chi_\pm{}^c\overline{\chi}_\pm{}^d\big], \tag{4.4g}$$

$$q_\mp \overline{\chi}_\pm{}^a = -i\Gamma_\pm{}^a{}_{bc}(x)\chi_\mp{}^b\overline{\chi}_\pm{}^c, \tag{4.4h}$$

$$\overline{q}_\pm \overline{\chi}_\pm{}^a = 0, \tag{4.4i}$$

$$\overline{q}_\mp \overline{\chi}_\pm{}^a = -i\Gamma_\pm{}^a{}_{bc}(x)\overline{\chi}_\mp{}^b\overline{\chi}_\pm{}^c. \tag{4.4j}$$

The $(2,2)$ supersymmetry of the sigma model action S is inherited by the quantum mechanics action S_{QM}, so that

$$q_\pm S_{QM} = \overline{q}_\pm S_{QM} = 0. \tag{4.5}$$

The associated four conserved supercharges $Q_\pm$, $\overline{Q}_\pm$ can be computed by the Noether procedure by letting the supersymmetry parameters $\alpha^\pm$, $\tilde{\alpha}^\pm$ to be time dependent:

$$\delta_S S_{QM} = \int_T dt\, i\big[\partial_t\alpha^+ Q_+ + \partial_t\tilde{\alpha}^+\overline{Q}_+ + \partial_t\alpha^- Q_- + \partial_t\tilde{\alpha}^-\overline{Q}_-\big]. \tag{4.6}$$

In this way, one finds that

$$Q_\pm = g_{ab}(x)\chi_\pm{}^a\partial_t x^b \mp \frac{i}{2}H_{abc}(x)\chi_\pm{}^a\chi_\pm{}^b\overline{\chi}_\pm{}^c, \tag{4.7a}$$

$$\overline{Q}_\pm = g_{ab}(x)\overline{\chi}_\pm{}^a\partial_t x^b \mp \frac{i}{2}H_{abc}(x)\overline{\chi}_\pm{}^a\overline{\chi}_\pm{}^b\chi_\pm{}^c. \tag{4.7b}$$

Similarly, the R symmetry variations can be read off from (3.11). It is convenient to decompose R variation operator δ_R as

$$\delta_R = i\big(\epsilon_V f_V + \epsilon_A f_A\big), \tag{4.8}$$

where the bosonic variation operators f_V, f_a are given by

$$f_V x^a = 0, \tag{4.9a}$$

$$f_A x^a = 0, \tag{4.9b}$$

$$f_V \chi_\pm{}^a = +\chi_\pm{}^a, \tag{4.9c}$$

$$f_A \chi_\pm{}^a = \pm\chi_\pm{}^a, \tag{4.9d}$$

$$f_V \overline{\chi}_\pm{}^a = -\overline{\chi}_\pm{}^a, \tag{4.9e}$$

$$f_A \overline{\chi}_\pm{}^a = \mp\overline{\chi}_\pm{}^a. \tag{4.9f}$$

The R symmetry of the sigma model action S is inherited by the quantum mechanics action S_{QM}, so that one has

$$f_V S_{QM} = f_A S_{QM} = 0. \tag{4.10}$$

The associated two conserved R charges F_V, F_A can be computed easily again by the Noether procedure by letting the R Symmetry parameters ϵ_V, ϵ_A to be time dependent:

$$\delta_R S_{QM} = -\int_T dt \big[\partial_t \epsilon_V F_V + \partial_t \epsilon_A F_A\big]. \tag{4.11}$$

In this way, one finds that

$$F_V = g_{ab}(x)\big(\overline{\chi}_+{}^a \chi_+{}^b + \overline{\chi}_-{}^a \chi_-{}^b\big), \tag{4.12a}$$

$$F_A = g_{ab}(x)\big(\overline{\chi}_+{}^a \chi_+{}^b - \overline{\chi}_-{}^a \chi_-{}^b\big). \tag{4.12b}$$

It is straightforward to see that

$$q_\pm{}^2 \approx 0, \tag{4.13a}$$

$$q_\pm q_\mp + q_\mp q_\pm \approx 0, \tag{4.13b}$$

$$\overline{q}_\pm{}^2 \approx 0, \tag{4.13c}$$

$$\overline{q}_\pm \overline{q}_\mp + \overline{q}_\mp \overline{q}_\pm \approx 0, \tag{4.13d}$$

$$q_\pm \overline{q}_\pm + \overline{q}_\pm q_\pm \approx -i\partial_t, \tag{4.13e}$$

$$q_\pm \overline{q}_\mp + \overline{q}_\mp q_\pm \approx 0, \tag{4.13f}$$

$$f_V q_\pm - q_\pm f_V \approx +q_\pm, \tag{4.13g}$$

$$f_A q_\pm - q_\pm f_A \approx \pm q_\pm, \tag{4.13h}$$

$$f_V \overline{q}_\pm - \overline{q}_\pm f_V \approx -\overline{q}_\pm, \tag{4.13i}$$

$$f_A \overline{q}_\pm - \overline{q}_\pm f_A \approx \mp \overline{q}_\pm, \tag{4.13j}$$

$$f_V f_A - f_A f_V \approx 0. \tag{4.13k}$$

where $\approx$ denotes equality on shell. In more precise terms, this means the following. Let $\mathscr{F}$ denote the algebra of local composite fields and let $\mathscr{E}$ be the ideal of

$\mathscr{F}$ generated by the field equations. Then, it can be shown that $q_\pm$, $\overline{q}_\pm$, f_V, f_A leave $\mathscr{E}$ invariant. In this way, $q_\pm$, $\overline{q}_\pm$, f_V, f_A define variations operators on the quotient algebra $\mathscr{F}_{\mathscr{E}} = \mathscr{F}/\mathscr{E}$. As such, they satisfy the algebra (4.13) with the on shell equality sign $\approx$ replaced by the usual equality sign $=$.

The above analysis shows that biHermitian supersymmetric quantum mechanics enjoys a $N = 4$ supersymmetry. If the target space is endowed with a structure containing the given biHermitian structure as a substructure, e. g. a hyperKaehler structure, the amount of supersymmetry may be enhanced [32]. We shall not explore this possibility in this paper.

5 Quantization

From the expression of the action S_{QM} of biHermitian supersymmetric quantum mechanics, eq. (4.1), one can easily read off the classical Lagrangian

$$L_{QM} = \frac{1}{2} g_{ab}(x) \partial_t x^a \partial_t x^b + \frac{i}{2} g_{ab}(x) \big(\overline{\chi}_+{}^a \nabla_{+t} \chi_+{}^b - \nabla_{+t} \overline{\chi}_+{}^a \chi_+{}^b \\ + \overline{\chi}_-{}^a \nabla_{-t} \chi_-{}^b - \nabla_{-t} \overline{\chi}_-{}^a \chi_-{}^b \big) + R_{+abcd}(x) \overline{\chi}_+{}^a \chi_+{}^b \overline{\chi}_-{}^c \chi_-{}^d. \tag{5.1}$$

In order L_{QM} to be real, the kinetic term of the fermion coordinates $\chi_\pm{}^a$, $\overline{\chi}_\pm{}^a$ in (4.1) has been cast in symmetric form by adding a total time derivative term.

The conjugate momenta of the coordinates x^a, $\chi_\pm{}^a$, $\overline{\chi}_\pm{}^a$ are defined

$$\pi_a = \frac{\partial L_{QM}}{\partial \partial_t x^a}, \tag{5.2a}$$

$$\lambda_{\pm a} = - \frac{\partial L_{QM}}{\partial \nabla_{\pm t} \chi_\pm{}^a}, \tag{5.2b}$$

$$\overline{\lambda}_{\pm a} = + \frac{\partial L_{QM}}{\partial \nabla_{\pm t} \overline{\chi}_\pm{}^a}. \tag{5.2c}$$

To have manifest covariance, we define the fermionic momenta by differentiating with respect to $\nabla_{\pm t} \chi_\pm{}^a$, $\nabla_{\pm t} \overline{\chi}_\pm{}^a$ rather than $\partial_t \chi_\pm{}^a$, $\partial_t \overline{\chi}_\pm{}^a$. In this way, the implicit dependence of $\nabla_{\pm t} \chi_\pm{}^a$, $\nabla_{\pm t} \overline{\chi}_\pm{}^a$ on $\partial_t x^a$ is disregarded in the computation of π_a. Therefore, the momenta π_a are not canonical. The advantages of this way of proceeding will become clear in due course. Explicitly, one has

$$\pi_a = g_{ab}(x) \partial_t x^b, \tag{5.3a}$$

$$\lambda_{\pm a} = \frac{i}{2} g_{ab}(x) \overline{\chi}_\pm{}^b, \tag{5.3b}$$

$$\overline{\lambda}_{\pm a} = -\frac{i}{2} g_{ab}(x) \chi_\pm{}^b. \tag{5.3c}$$

Note that the constraints $\overline{\Lambda}_\pm{}^a{}_b(x) \chi_\pm{}^b = \Lambda_\pm{}^a{}_b(x) \overline{\chi}_\pm{}^b = 0$ imply correspondingly the constraints $\overline{\Lambda}_\pm{}^b{}_a(x) \lambda_{\pm b} = \Lambda_\pm{}^b{}_a(x) \overline{\lambda}_{\pm b} = 0$.

The classical Hamiltonian is computed as usual

$$H_{QM} = \pi_a \partial_t x^a + \lambda_{+a} \nabla_{+t} \chi_+{}^a + \lambda_{-a} \nabla_{-t} \chi_-{}^a \\ - \overline{\lambda}_{+a} \nabla_{+t} \overline{\chi}_+{}^a - \overline{\lambda}_{-a} \nabla_{-t} \overline{\chi}_-{}^a - L_{QM}. \tag{5.4}$$

The resulting expression of H_{QM} is

$$H_{QM} = \frac{1}{2} g^{ab}(x)\pi_a\pi_b - R_{+abcd}(x)\overline{\chi}_+{}^a\chi_+{}^b\overline{\chi}_-{}^c\chi_-{}^d. \tag{5.5}$$

The graded Poisson brackets of the coordinates x^a, $\chi_\pm{}^a$, $\overline{\chi}_\pm{}^a$ and momenta π_a, $\lambda_{\pm a}$, $\overline{\lambda}_{\pm a}$ are given by

$$\{x^a, \pi_b\}_P = \delta^a{}_b, \tag{5.6a}$$

$$\begin{aligned}\{\pi_a, \pi_b\}_P = R_+{}^c{}_{dab}(x)(\lambda_{+c}\chi_+{}^d - \overline{\lambda}_{+c}\overline{\chi}_+{}^d) \\ + R_-{}^c{}_{dab}(x)(\lambda_{-c}\chi_-{}^d - \overline{\lambda}_{-c}\overline{\chi}_-{}^d),\end{aligned} \tag{5.6b}$$

$$\{\pi_a, \chi_\pm{}^b\}_P = \Gamma_\pm{}^b{}_{ac}(x)\chi_\pm{}^c, \tag{5.6c}$$

$$\{\pi_a, \overline{\chi}_\pm{}^b\}_P = \Gamma_\pm{}^b{}_{ac}(x)\overline{\chi}_\pm{}^c, \tag{5.6d}$$

$$\{\pi_a, \lambda_{\pm b}\}_P = -\Gamma_\pm{}^c{}_{ab}(x)\lambda_{\pm c}, \tag{5.6e}$$

$$\{\pi_a, \overline{\lambda}_{\pm b}\}_P = -\Gamma_\pm{}^c{}_{ab}(x)\overline{\lambda}_{\pm c}, \tag{5.6f}$$

$$\{\chi_\pm{}^a, \lambda_{\pm b}\}_P = \Lambda_\pm{}^a{}_b(x), \tag{5.6g}$$

$$\{\overline{\chi}_\pm{}^a, \overline{\lambda}_{\pm b}\}_P = -\overline{\Lambda}_\pm{}^a{}_b(x), \tag{5.6h}$$

all remaining Poisson brackets vanishing identically. The form of the brackets (5.6c)–(5.6f) is dictated by covariance and the constraints $\overline{\Lambda}_\pm{}^a{}_b(x)\chi_\pm{}^b = \Lambda_\pm{}^a{}_b(x)\overline{\chi}_\pm{}^b = 0$, $\overline{\Lambda}_\pm{}^b{}_a(x)\lambda_{\pm b} = \Lambda_\pm{}^b{}_a(x)\overline{\lambda}_{\pm b} = 0$. The form of the Poisson bracket (5.6b) is then essentially determined by the fulfillment of the Jacobi identity.

From (5.3b), (5.3c), it follows that the constraints

$$C_{\pm a} := \lambda_{\pm a} - \frac{i}{2} g_{ab}(x)\overline{\chi}_\pm{}^b \simeq 0, \tag{5.7a}$$

$$\overline{C}_{\pm a} := \overline{\lambda}_{\pm a} + \frac{i}{2} g_{ab}(x)\chi_\pm{}^b \simeq 0 \tag{5.7b}$$

hold, where $\simeq$ denotes weak equality in Dirac's sense. These constraints are second class, as follows from the Poisson brackets

$$\{C_{\pm a}, \overline{C}_{\pm b}\}_P = i\Lambda_{\pm ab}(x), \tag{5.8}$$

all remaining Poisson brackets of the constraints vanishing. The resulting graded Dirac brackets of the independent variables x^a, π_a, $\chi_\pm{}^a$, $\overline{\chi}_\pm{}^a$ are easily computed:

$$\{x^a, \pi_b\}_D = \delta^a{}_b, \tag{5.9a}$$

$$\{\pi_a, \pi_b\}_D = iR_{+cdab}(x)\overline{\chi}_+{}^c\chi_+{}^d + iR_{-cdab}(x)\overline{\chi}_-{}^c\chi_-{}^d, \tag{5.9b}$$

$$\{\pi_a, \chi_\pm{}^b\}_D = \Gamma_\pm{}^b{}_{ac}(x)\chi_\pm{}^c, \tag{5.9c}$$

$$\{\pi_a, \overline{\chi}_\pm{}^b\}_D = \Gamma_\pm{}^b{}_{ac}(x)\overline{\chi}_\pm{}^c, \tag{5.9d}$$

$$\{\chi_\pm{}^a, \overline{\chi}_\pm{}^b\}_D = -i\Lambda_\pm{}^{ab}(x), \tag{5.9e}$$

all remaining Dirac brackets vanishing identically.

To quantize the theory, we promote the variables x^a, π_a, $\chi_\pm{}^a$, $\overline{\chi}_\pm{}^a$ to operators and stipulate that their graded commutators are given by the formal substitution $\{\,,\,\}_D \rightarrow -i[\,,\,]$. In the case of the Dirac bracket (5.9b), there is an obvious ordering problem. The choice of ordering given below is the only one that is compatible with the supersymmetry algebra at the quantum level, eq. (6.1), as will be shown in the next section. In this way, we obtain

$$[x^a, \pi_b] = i\delta^a{}_b, \tag{5.10a}$$

$$\begin{aligned}[\pi_a, \pi_b] = &-\frac{1}{2}R_{+cdab}(x)(\overline{\chi}_+{}^c\chi_+{}^d - \chi_+{}^d\overline{\chi}_+{}^c) \\ &-\frac{1}{2}R_{-cdab}(x)(\overline{\chi}_-{}^c\chi_-{}^d - \chi_-{}^d\overline{\chi}_-{}^c),\end{aligned} \tag{5.10b}$$

$$[\pi_a, \chi_\pm{}^b] = i\Gamma_\pm{}^b{}_{ac}(x)\chi_\pm{}^c, \tag{5.10c}$$

$$[\pi_a, \overline{\chi}_\pm{}^b] = i\Gamma_\pm{}^b{}_{ac}(x)\overline{\chi}_\pm{}^c, \tag{5.10d}$$

$$[\chi_\pm{}^a, \overline{\chi}_\pm{}^b] = \Lambda_\pm{}^{ab}(x), \tag{5.10e}$$

all remaining commutators vanishing. The commutation relations are compatible with the Jacobi identities, as is easy to check. As to covariance, under a change of target space coordinates $t^a \rightarrow t'^a$, the operators x^a, π_a, $\chi_\pm{}^a$, $\overline{\chi}_\pm{}^a$ behave as their classical counterparts. For the operator π_a, there is again an ordering problem. It can be seen that the ordering of the coordinate transformation relation of π_a compatible with covariance is

$$\pi'_a = \frac{\partial t^b}{\partial t'^a}(x)\pi_b. \tag{5.11}$$

Assuming this, the commutation relations (5.10) are straightforwardly checked to be covariant, as required.

As explained earlier, the above quantization prescription is manifestly covariant but not canonical. When studying Hilbert space representations of the quantum operator algebra, it may be convenient to have a canonical quantization prescription. To construct this, let us define

$$\epsilon^a = \frac{1}{2^{\frac{1}{2}}}\big(\chi_+{}^a + \overline{\chi}_+{}^a + i\chi_-{}^a + i\overline{\chi}_-{}^a\big), \tag{5.12a}$$

$$\iota_a = \frac{1}{2^{\frac{1}{2}}} g_{ab}(x)\big(\chi_+{}^b + \overline{\chi}_+{}^b - i\chi_-{}^b - i\overline{\chi}_-{}^b\big), \tag{5.12b}$$

and set

$$p_a = \pi_a - i\Gamma^c{}_{ab}(x)\epsilon^b\iota_c - \frac{i}{4}H_{abc}(x)\epsilon^b\epsilon^c - \frac{i}{4}H_a{}^{bc}(x)\iota_b\iota_c. \tag{5.13}$$

Using (5.10), it is straightforward to verify that x^a, p_a, ϵ^a, ι_a satisfy the quantum graded commutation relations

$$[x^a, p_b] = i\delta^a{}_b, \tag{5.14a}$$

$$[\epsilon^a, \iota_b] = \delta^a{}_b, \tag{5.14b}$$

all other commutators vanishing. Note that, in particular, $[p_a, p_b] = 0$, while $[\pi_a, \pi_b] \neq 0$. Thus, unlike the π_a, the momenta p_a are canonical, as required.

Another issue related to quantization is that of the Hermiticity properties of the operators x^a, π_a, $\chi_\pm{}^a$, $\overline{\chi}_\pm{}^a$. However, this problem cannot be posed at the level of target space local coordinate representations of these operators in general. Hermiticity is essentially a target space global property, since the Hilbert space product involves an integration over M. This is obvious also from the coordinate transformation relation (5.11), which are not compatible with a naive Hermiticity relation of the form $\pi_a{}^* = \pi_a$. Similar considerations apply to the canonical operators x^a, p_a, ϵ^a, ι_a.

This completes the quantization of biHermitian supersymmetric quantum mechanics. In the Kaehler case, similar results were obtained in [35, 36].

6 The quantum symmetry algebra

As shown in section 4, biHermitian supersymmetric quantum mechanics has a rich symmetry structure which should be reproduced at the quantum level. This leads to the requirement that the graded commutation relations

$$[Q_\pm, Q_\pm] = 0, \tag{6.1a}$$

$$[Q_\pm, Q_\mp] = 0, \tag{6.1b}$$

$$[\overline{Q}_\pm, \overline{Q}_\pm] = 0, \tag{6.1c}$$

$$[\overline{Q}_\pm, \overline{Q}_\mp] = 0, \tag{6.1d}$$

$$[Q_\pm, \overline{Q}_\pm] = H_{QM}, \tag{6.1e}$$

$$[Q_\pm, \overline{Q}_\mp] = 0, \tag{6.1f}$$

$$[F_V, Q_\pm] = -Q_\pm, \tag{6.1g}$$

$$[F_A, Q_\pm] = \mp Q_\pm, \tag{6.1h}$$

$$[F_V, \overline{Q}_\pm] = +\overline{Q}_\pm, \tag{6.1i}$$

$$[F_A, \overline{Q}_\pm] = \pm \overline{Q}_\pm, \tag{6.1j}$$

$$[F_V, F_A] = 0 \tag{6.1k}$$

should hold upon quantization. This symmetry algebra is in fact the quantum counterpart of the algebra (4.13) obeyed by the classical variation operators $q_\pm$, $\overline{q}_\pm$, f_V, f_A defined in (4.4), (4.9). Demanding that the operator relations (6.1) hold is a very stringent requirement. It is not obvious a priori that a quantization of the theory compatible with (6.1) exists, but in fact it does and it is unique. Indeed, imposing (6.1) determines not only all ordering ambiguities of the commutators of the basic operator variables x^a, π_a, ${\chi_\pm}^a$, ${\overline{\chi}_\pm}^a$, eqs. (5.10), as anticipated in the previous section, but also those of the supercharges $Q_\pm$, $\overline{Q}_\pm$ and the Hamiltonian H_{QM}. It does not determine conversely the ordering ambiguities of the vector and axial R charges F_V, F_A. However, these ambiguities amount to a harmless additive c number that can be fixed conventionally, as can be easily checked.

In this way, upon quantization, one finds that $Q_\pm$, $\overline{Q}_\pm$ are given by

$$Q_\pm = \chi_\pm{}^a\pi_a \mp \frac{i}{2}H_{abc}(x)\chi_\pm{}^a\chi_\pm{}^b\overline{\chi}_\pm{}^c \pm \frac{i}{2}H_{abc}\Lambda_\pm{}^{bc}(x)\chi_\pm{}^a, \tag{6.2a}$$

$$\overline{Q}_\pm = \overline{\chi}_\pm{}^a\pi_a \mp \frac{i}{2}H_{abc}(x)\overline{\chi}_\pm{}^a\overline{\chi}_\pm{}^b\chi_\pm{}^c \pm \frac{i}{2}H_{abc}\overline{\Lambda}_\pm{}^{bc}(x)\overline{\chi}_\pm{}^a. \tag{6.2b}$$

The last term in the right hand side of both relations is a quantum ordering effect. Likewise, the quantum Hamiltonian is

$$\begin{aligned} H_{QM} = &\frac{1}{2}\pi_a g^{ab}(x)\pi_b - \frac{i}{2}\Gamma^a{}_{ac}g^{cb}(x)\pi_b \\ &-\frac{1}{4}R_{+abcd}(x)(\overline{\chi}_+{}^a\chi_+{}^b - \chi_+{}^b\overline{\chi}_+{}^a)(\overline{\chi}_-{}^c\chi_-{}^d - \chi_-{}^d\overline{\chi}_-{}^c) \\ &+\frac{1}{8}R(x) - \frac{1}{96}H^{abc}H_{abc}(x). \end{aligned} \tag{6.3}$$

Here, $\Gamma^a{}_{bc}$ is the usual Levi Civita connection and R is the corresponding Ricci scalar. The second, fourth and fifth term in the right hand side are again ordering effects. Finally, the R charges are given by

$$F_V = g_{ab}(x)\big(\overline{\chi}_+{}^a\chi_+{}^b + \overline{\chi}_-{}^a\chi_-{}^b\big) - \frac{n}{2}, \tag{6.4a}$$

$$F_A = g_{ab}(x)\big(\overline{\chi}_+{}^a\chi_+{}^b - \overline{\chi}_-{}^a\chi_-{}^b\big), \tag{6.4b}$$

where $n = \dim_{\mathbb{R}} M$, with a conventional choice of operator ordering.

One can express the supercharges $Q_\pm$, $\overline{Q}_\pm$, the Hamiltonian H_{QM} and the R charges F_V, F_A in terms of the canonical operators x^a, p_a, ϵ^a, ι_a. From (6.2), the supercharges are given by

$$Q_\pm = \Lambda_\pm{}^b{}_a(x)\psi_\pm{}^a P_{\mp b} \pm \frac{i}{2}H_{abc}\Lambda_\pm{}^{bc}(x)\psi_\pm{}^a \pm \frac{i}{6}H_{abc}(x)\psi_\pm{}^a\psi_\pm{}^b\psi_\pm{}^c, \tag{6.5a}$$

$$\overline{Q}_\pm = \overline{\Lambda}_\pm{}^b{}_a(x)\psi_\pm{}^a P_{\mp b} \pm \frac{i}{2}H_{abc}\overline{\Lambda}_\pm{}^{bc}(x)\psi_\pm{}^a \pm \frac{i}{6}H_{abc}(x)\psi_\pm{}^a\psi_\pm{}^b\psi_\pm{}^c, \tag{6.5b}$$

where $\psi_\pm{}^a$, $P_{\pm a}$ are given by

$$\psi_+{}^a = \frac{1}{2^{\frac{1}{2}}}\big(\epsilon^a + g^{ab}(x)\iota_b\big), \tag{6.6a}$$

$$\psi_-{}^a = \frac{1}{2^{\frac{1}{2}}i}\big(\epsilon^a - g^{ab}(x)\iota_b\big) \tag{6.6b}$$

and

$$P_{\pm a} = p_a + i\Gamma_{\pm}{}^c{}_{ab}(x)\epsilon^b\iota_c. \tag{6.7}$$

From (6.3), the quantum Hamiltonian is

$$\begin{aligned} H_{QM} = & \frac{1}{2}g^{ab}(x)\pi_a\pi_b + \frac{i}{2}\Gamma^a{}_{bc}g^{bc}(x)\pi_a - \frac{1}{4}R_{+abcd}(x)\psi_+{}^a\psi_+{}^b\psi_-{}^c\psi_-{}^d \\ & + \frac{1}{8}R(x) - \frac{1}{96}H^{abc}H_{abc}(x), \end{aligned} \tag{6.8}$$

where π_a is given by

$$\pi_a = p_a + i\Gamma^c{}_{ab}(x)\epsilon^b\iota_c + \frac{i}{4}H_{abc}(x)\epsilon^b\epsilon^c + \frac{i}{4}H_a{}^{bc}(x)\iota_b\iota_c. \tag{6.9}$$

Finally, from (6.4), the R charges are given by

$$F_V = \Lambda_{+ab}(x)\psi_+{}^a\psi_+{}^b + \Lambda_{-ab}(x)\psi_-{}^a\psi_-{}^b - \frac{n}{2}, \tag{6.10a}$$

$$F_A = \Lambda_{+ab}(x)\psi_+{}^a\psi_+{}^b - \Lambda_{-ab}(x)\psi_-{}^a\psi_-{}^b. \tag{6.10b}$$

In this way, we succeeded in quantizing biHermitian supersymmetric quantum mechanics in a way compatible with supersymmetry and R symmetry. The next step of our analysis would be the search for interesting Hilbert space representations of the operator algebra. We shall postpone this to the next section. Here, we shall analyze which properties a representation should have on general physical grounds.

A representation of the operator algebra in a Hilbert space $\mathcal{V}$ allows one to define the adjoint of any operator. The Hamiltonian H_{QM} of biHermitian supersymmetric quantum mechanics should be selfadjoint

$$H_{QM}{}^* = H_{QM}, \tag{6.11}$$

to have a real energy spectrum. From (6.11), on account of (6.1e), the supercharges $Q_\pm$, $\overline{Q}_\pm$ should satisfy

$$Q_\pm{}^* = \overline{Q}_\pm \tag{6.12}$$

under adjunction. The R charges should also be selfadjoint

$$F_V{}^* = F_V, \tag{6.13a}$$

$$F_A{}^* = F_A, \tag{6.13b}$$

to have a real R charge spectrum.

From (6.1k), (6.13), it follows that that the R charges F_V, F_A can simultaneously be diagonalized in $\mathcal{V}$. Since F_V, F_A are the infinitesimal generators of the $U(1)_V$, $U(1)_A$ symmetry groups, they should satisfy the conditions

$$\exp(2\pi i F_V) = 1, \tag{6.14a}$$

$$\exp(2\pi i F_A) = 1. \tag{6.14b}$$

This implies that the spectra of F_V, F_A are subsets of $\mathbb{Z}$. It follows that the Hilbert space $\mathcal{V}$ has a direct sum decomposition of the form

$$\mathcal{V} = \bigoplus_{k_V, k_A \in \mathbb{Z}} \mathcal{V}_{k_V, k_A}, \tag{6.15}$$

where $\mathcal{V}_{k_V,k_A}$ is the joint eigenspace of F_V, F_A of eigenvalues k_V, k_A.

Only a finite number of the subspaces $\mathcal{V}_{k_V,k_A}$ are non zero. A technical analysis outlined in appendix B shows that $\mathcal{V}_{k_V,k_A} = 0$ unless the conditions

$$-\frac{n}{2} \leq k_V \pm k_A \leq \frac{n}{2}, \tag{6.16a}$$

$$k_V \pm k_A = \frac{n}{2} \mod 2 \tag{6.16b}$$

are simultaneously satisfied. The (6.16) imply that

$$-\frac{n}{2} \leq k_V, k_A \leq \frac{n}{2} \tag{6.17}$$

and further, that

$$-\frac{n}{2} + |k_V| \leq k_A \leq \frac{n}{2} - |k_V|, \tag{6.18a}$$

$$-\frac{n}{2} + |k_A| \leq k_V \leq \frac{n}{2} - |k_A|, \tag{6.18b}$$

as is easy to see. Moreover, for fixed k_V (k_A), two consecutive eigenvalues k_A (k_V) differ precisely by 2 units. Thus, the non vanishing $\mathcal{V}_{k_V,k_A}$ can be arranged in a diamond shaped array as follows

$$\begin{array}{ccccccc} & & & \mathcal{V}_{0,n/2} & & & \\ & & \cdots & & \cdots & & \\ & \mathcal{V}_{-n/2+1,1} & & & & \mathcal{V}_{n/2-1,1} & \\ \mathcal{V}_{-n/2,0} & & & \cdots & & & \mathcal{V}_{n/2,0} \\ & \mathcal{V}_{-n/2+1,-1} & & & & \mathcal{V}_{n/2-1,-1} & \\ & & \cdots & & \cdots & & \\ & & & \mathcal{V}_{0,-n/2} & & & \end{array} . \tag{6.19}$$

Using the the commutation relations (6.1g)–(6.1j), it is readily verified that the supercharges $Q_\pm$, $\overline{Q}_\pm$ act as follows

$$\begin{tikzcd} \mathcal{V}_{k_V-1,k_A+1} & & \mathcal{V}_{k_V+1,k_A+1} \\ & \mathcal{V}_{k_V,k_A} \arrow[ul, "Q_-"'] \arrow[ur, "\overline{Q}_+"] \arrow[dl, "Q_+"] \arrow[dr, "\overline{Q}_-"'] & \\ \mathcal{V}_{k_V-1,k_A-1} & & \mathcal{V}_{k_V+1,k_A-1} \end{tikzcd} . \tag{6.20}$$

From (6.1e), (6.20), it follows easily that the spaces $\mathcal{V}_{k_V,k_A}$ are invariant under the Hamiltonian H_{QM}. Each space $\mathcal{V}_{k_V,k_A}$ contains a subspace ${\mathcal{V}^{(0)}}_{k_V,k_A}$ spanned by the zero energy states $|\alpha\rangle \in \mathcal{V}_{k_V,k_A}$,

$$H_{QM}|\alpha\rangle = 0. \tag{6.21}$$

By (6.1e), these are precisely the supersymmetric states $|\alpha\rangle \in \mathcal{V}_{k_V,k_A}$,

$$Q_\pm|\alpha\rangle = \overline{Q}_\pm|\alpha\rangle = 0. \tag{6.22}$$

By (6.1a), (6.1c), the supercharges $Q_\pm$, $\overline{Q}_\pm$ are nilpotent and, so, are characterized by the cohomology spaces $H^{k_V,k_A}(Q_\pm) = \ker Q_\pm \cap \mathcal{V}_{k_V,k_A} / \operatorname{im} Q_\pm \cap \mathcal{V}_{k_V,k_A}$, $H^{k_V,k_A}(\overline{Q}_\pm) = \ker \overline{Q}_\pm \cap \mathcal{V}_{k_V,k_A} / \operatorname{im} \overline{Q}_\pm \cap \mathcal{V}_{k_V,k_A}$. By standard arguments of supersymmetric quantum mechanics, the $H^{k_V,k_A}(Q_\pm)$, $H^{k_V,k_A}(\overline{Q}_\pm)$ are all isomorphic to ${\mathcal{V}^{(0)}}_{k_V,k_A}$: each state of ${\mathcal{V}^{(0)}}_{k_V,k_A}$ represents a distinct cohomology class of $Q_\pm$, $\overline{Q}_\pm$ and each cohomology class of $Q_\pm$, $\overline{Q}_\pm$ has a unique representative state in ${\mathcal{V}^{(0)}}_{k_V,k_A}$.

Consider the total supercharge

$$Q = Q_+ + \overline{Q}_+ + iQ_- + i\overline{Q}_- \tag{6.23}$$

and its adjoint

$$Q^* = Q_+ + \overline{Q}_+ - iQ_- - i\overline{Q}_-. \tag{6.24}$$

Q, Q^* satisfy the graded commutation relations

$$[Q, Q] = 0, \tag{6.25a}$$

$$[Q^*, Q^*] = 0, \tag{6.25b}$$

$$[Q, Q^*] = 4H_{QM}, \tag{6.25c}$$

It follows from here that the supercharges Q, Q^* are nilpotent and that their cohomology spaces $H^{k_V,k_A}(Q)$, $H^{k_V,k_A}(Q^*)$ are both isomorphic to ${\mathcal{V}^{(0)}}_{k_V,k_A}$. Thus, the common cohomology of the supercharges $Q_\pm$, $\overline{Q}_\pm$ can be described in terms of the total supercharges Q, Q^*. This description is more transparent. Indeed, using (6.5), (6.6), (6.7), it is straightforward to verify that Q, Q^* are given by

$$Q = -2^{\frac{1}{2}}i\big[\epsilon^a ip_a - \frac{1}{6}H_{abc}(x)\epsilon^a\epsilon^b\epsilon^c\big], \tag{6.26a}$$

$$Q^* = +2^{\frac{1}{2}}i\big[-g^{ab}(x)\iota_a\big(ip_b - {\Gamma^d}_{bc}(x)\epsilon^c\iota_d\big) + \frac{1}{6}H^{abc}(x)\iota_a\iota_b\iota_c\big]. \tag{6.26b}$$

From here, it appears that Q depends on the 3–form H but it does not depend on metric g and the complex structures $K^\pm$ for given H. So does its cohomology. This fact was noticed long ago in reference [37].

7 The differential form representation

As in Riemannian supersymmetric quantum mechanics, the quantum operator algebra of biHermitian supersymmetric quantum mechanics has a representation by operators acting on a space of inhomogeneous complex differential forms.

We assume first that the target manifold M is compact. This ensures convergence of integration over M. The anticommutator algebra (5.14b) of the canonical fermion operators ϵ^a, ι_a has the well–known form of a fermionic creation and annihilation algebra. Therefore, it admits a standard Fock space representation. The Fock vacuum $|0\rangle$ is defined as usual by

$$\iota_a|0\rangle = 0. \tag{7.1}$$

The most general state vector is of the form

$$|\omega\rangle = \sum_{p=0}^{n} \frac{1}{p!}\omega^{(p)}{}_{a_1\ldots a_p}(x)\epsilon^{a_1}\ldots\epsilon^{a_p}|0\rangle, \tag{7.2}$$

where the $\omega^{(p)} \in \Omega^p(M)\otimes\mathbb{C}$ are arbitrary complex p–forms. Therefore, there is a one–to–one correspondence between state vectors and inhomogeneous differential forms. The vacuum itself corresponds to the constant 0–form 1,

$$|0\rangle := 1. \tag{7.3}$$

In this way, the action of the operators x^a, p_a, ϵ^a, ι_a on the state vector $|\omega\rangle$ is represented by operators acting on the space of inhomogeneous complex differential forms $\mathcal{V} = \Omega^*(M)\otimes\mathbb{C}$ according to the prescription

$$x^a := t^a\cdot, \tag{7.4a}$$

$$p_a := -i\partial/\partial t^a, \tag{7.4b}$$

$$\epsilon^a := dt^a\wedge, \tag{7.4c}$$

$$\iota_a := \iota_{\partial/\partial t^a}, \tag{7.4d}$$

where t^a is a local coordinate. This yields the differential form representation of biHermitian supersymmetric quantum mechanics. In this representation, the supercharges $Q_\pm$, $\overline{Q}_\pm$ and the R charges F_V, F_A are given by first and zeroth order differential operators on $\mathcal{V}$, respectively.

In the differential form representation, the Hilbert space product is given by the usual formula

$$\langle\alpha|\beta\rangle = \frac{1}{\mathrm{vol}(M)}\int_M d^n t\, g^{\frac{1}{2}} \sum_p \frac{1}{p!}\alpha^{(p)a_1\ldots a_p}\beta^{(p)}{}_{a_1\ldots a_p}, \tag{7.5}$$

with $\alpha, \beta \in \mathcal{V}$. This allows to define the adjoint of the relevant operators. It is straightforward to check that the supercharges $Q_\pm$, $\overline{Q}_\pm$, the Hamiltonian H_{QM} and the R charges F_V, F_A satisfy the adjunction relations (6.12), (6.11), (6.13), respectively, as required.

If M is not compact, we can repeat the above construction with a few changes. It is necessary to restrict to differential forms with compact support. The state space is therefore isomorphic to $\mathcal{V}_c = \Omega_c{}^*(M) \otimes \mathbb{C}$. The Hilbert space product is given again by (7.5) with the prefactor $1/\mathrm{vol}(M)$ removed and $\alpha, \beta \in \mathcal{V}_c$. Note that the Fock vacuum $|0\rangle$ is no longer normalized.

8 Relation to generalized Hodge theory

The algebraic framework described in the second half of section 6 is very reminiscent of the Hodge theory of compact generalized Kaehler manifolds developed by Gualtieri in [33] and reviewed briefly below [28, 34]. Indeed, in the differential form representation of section 7, they coincide, as we show below. This result, besides being interesting in its own, sheds also light on the nature of the space of supersymmetric ground states of biHermitian supersymmetric quantum mechanics. In what follows, we assume that M is compact.

The bundle $TM \oplus T^*M$ is endowed with a canonical indefinite metric. The Clifford bundle $C\ell(TM \oplus T^*M)$ is thus defined. The space of spinor fields of $C\ell(TM \oplus T^*M)$ is precisely the space $\mathcal{V} = \Omega^*(M) \otimes \mathbb{C}$. The Clifford action of a section $X + \xi$ of $TM \oplus T^*M$ is defined by

$$(X + \xi)\cdot = X^a(x)\iota_a + \xi_a(x)\epsilon^a, \tag{8.1}$$

where (7.4) holds. The biHermitian data $(g, H, K_\pm)$ are codified in two commuting H-twisted generalized complex structures

$$\mathcal{J}_{1/2} = \frac{1}{2}\begin{pmatrix} K_+ \pm K_- & (K_+ \mp K_-)g^{-1} \\ g(K_+ \mp K_-) & -({K_+}^t \pm {K_-}^t) \end{pmatrix}. \tag{8.2}$$

defining a H-twisted generalized Kaehler structure [26]. $\mathcal{J}_{1/2}$ are sections of the bundle $\mathfrak{so}(TM \oplus T^*M)$. They act on $\mathcal{V}$ via the Clifford action,

$$\begin{aligned}\mathcal{J}_{1/2}\cdot = \frac{1}{2}\Big\{&\frac{1}{2}(K_+ \mp K_-)^{ab}(x)\iota_a\iota_b + \frac{1}{2}(K_+ \mp K_-)_{ab}(x)\epsilon^a\epsilon^b \\ &+ \frac{1}{2}(K_+ \pm K_-)^a{}_b(x)(\iota_a\epsilon^b - \epsilon^b\iota_a)\Big\}.\end{aligned} \tag{8.3}$$

It can be shown that the operators $\mathcal{J}_{1/2}\cdot$ commute and that their spectra are subsets of $i\mathbb{Z}$ [33]. In this way, $\mathcal{V}$ decomposes as a direct sum of joint eigenspaces ${\mathcal{V}'}_{k_1,k_2}$ of $\mathcal{J}_{1/2}\cdot$ labeled by two integers $k_1, k_2 \in \mathbb{Z}$,

$$\mathcal{V} = \bigoplus_{k_1,k_2\in\mathbb{Z}} {\mathcal{V}'}_{k_1,k_2}, \tag{8.4}$$

in analogy to (6.15). Further, the non vanishing subspaces ${\mathcal{V}'}_{k_1,k_2}$ can be arranged in a diamond shaped array

$$\begin{array}{ccccccc} & & & {\mathcal{V}'}_{0,n/2} & & & \\ & & \cdots & & \cdots & & \\ & {\mathcal{V}'}_{-n/2+1,1} & & & & {\mathcal{V}'}_{n/2-1,1} & \\ {\mathcal{V}'}_{-n/2,0} & & & \cdots & & & {\mathcal{V}'}_{n/2,0} \\ & {\mathcal{V}'}_{-n/2+1,-1} & & & & {\mathcal{V}'}_{n/2-1,-1} & \\ & & \cdots & & \cdots & & \\ & & & {\mathcal{V}'}_{0,-n/2} & & & \end{array} \tag{8.5}$$

analogous to (6.19).

The H twisted differential $d_H = d - H\wedge$ is given by

$$d_H = \epsilon^a i p_a - \frac{1}{6} H_{abc}(x)\epsilon^a\epsilon^b\epsilon^c, \tag{8.6}$$

where, again, (7.4) holds. In [33], it is shown that $d_H : {\mathcal{V}'}_{k_1,k_2} \to {\mathcal{V}'}_{k_1-1,k_2-1} \oplus {\mathcal{V}'}_{k_1-1,k_2+1} \oplus {\mathcal{V}'}_{k_1+1,k_2+1} \oplus {\mathcal{V}'}_{k_1+1,k_2-1}$. Therefore, projecting on the four direct summands, d_H decomposes as a sum of the form

$$d_H = \delta_+ + \delta_- + \overline{\delta}_+ + \overline{\delta}_-, \tag{8.7}$$

where the operators $\delta_\pm$, $\overline{\delta}_\pm$ act as

$$\begin{tikzcd} {\mathcal{V}'}_{k_1-1,k_2+1} & & {\mathcal{V}'}_{k_1+1,k_2+1} \\ & {\mathcal{V}'}_{k_1,k_2} \arrow[ul, "\delta_-"'] \arrow[ur, "\overline{\delta}_+"] \arrow[dl, "\delta_+"] \arrow[dr, "\overline{\delta}_-"'] & \\ {\mathcal{V}'}_{k_1-1,k_2-1} & & {\mathcal{V}'}_{k_1+1,k_2-1} \end{tikzcd} \tag{8.8}$$

in a way analogous to (6.20).

Now, using (6.6), (6.10), (8.3), it is straightforward to verify that

$$\mathcal{J}_1 \cdot = iF_V, \qquad \mathcal{J}_2 \cdot = iF_A. \tag{8.9}$$

Thus, one has $\mathcal{V}'_{k_1,k_2} = \mathcal{V}_{k_V,k_A}$ for $k_1 = k_V$, $k_2 = k_A$. So, the direct sum decomposition (6.15), (8.4) of $\mathcal{V}$ coincide. From (6.26a), (8.6), it appears that

$$d_H = -\frac{1}{2^{\frac{1}{2}}i} Q \tag{8.10}$$

where Q is the total supercharge (6.23). Comparing (8.7), (6.23) and taking (8.8), (6.20) into account leads immediately to the following identifications

$$\delta_+ = -\frac{1}{2^{\frac{1}{2}}i} Q_+, \tag{8.11a}$$

$$\overline{\delta}_+ = -\frac{1}{2^{\frac{1}{2}}i} \overline{Q}_+, \tag{8.11b}$$

$$\delta_- = -\frac{1}{2^{\frac{1}{2}}} Q_-, \tag{8.11c}$$

$$\overline{\delta}_- = -\frac{1}{2^{\frac{1}{2}}} \overline{Q}_-. \tag{8.11d}$$

Combining (6.12) and (8.11), one finds that $\delta_\pm{}^* = \mp\overline{\delta}_\pm$, relations in fact obtained in [33] by different methods. Relations (8.11) are the main results of this paper. (8.9), (8.10) were obtained in [9].

In [33], it is also shown that the Lie algebroid differentials $\overline{\partial}_i$ associated to the generalized complex structures $\mathcal{J}_i$ are given by

$$\overline{\partial}_1 = \overline{\delta}_+ + \overline{\delta}_-, \tag{8.12a}$$

$$\overline{\partial}_2 = \overline{\delta}_+ + \delta_-. \tag{8.12b}$$

From (8.11), one finds that the $\overline{\partial}_i$ are related to the supercharges $Q_\pm$, $\overline{Q}_\pm$ as

$$\overline{\partial}_1 = -\frac{1}{2^{\frac{1}{2}}i} Q_B, \tag{8.13a}$$

$$\overline{\partial}_2 = -\frac{1}{2^{\frac{1}{2}}i} Q_A, \tag{8.13b}$$

where the supercharges Q_B, Q_A are given by

$$Q_B = \overline{Q}_+ + i\overline{Q}_- \tag{8.14a}$$

$$Q_A = \overline{Q}_+ + iQ_- \tag{8.14b}$$

[3]. In [9], Q_B, Q_A were related to the BRST charges of the B, A topological biHermitian sigma models. We shall come back to this in section 9.

The correspondence between the operators of Hodge theory of generalized Kaehler geometry and those of biHermitian supersymmetric quantum mechanics is summarized by overlapping the following diagrams

$$\begin{tikzcd}
\mathcal{V}_{k_1-1,k_2+1} & k_2 & \mathcal{V}_{k_1+1,k_2+1} \\
{} & \mathcal{V}_{k_1,k_2} \arrow[ul, "\delta_-"'] \arrow[u, dotted, "\overline{\partial}_2"] \arrow[ur, "\overline{\delta}_+"] \arrow[r, dotted, "\overline{\partial}_1"] \arrow[dr, "\overline{\delta}_-"] \arrow[dl, "\delta_+"'] \arrow[l, dotted, "\partial_1"] \arrow[d, dotted, "\partial_2"] & k_1 \\
\mathcal{V}_{k_1-1,k_2-1} & {} & \mathcal{V}_{k_1+1,k_2-1}
\end{tikzcd} \tag{8.15a}$$

$$\begin{tikzcd}
\mathcal{V}_{k_V-1,k_A+1} & k_A & \mathcal{V}_{k_V+1,k_A+1} \\
{} & \mathcal{V}_{k_V,k_A} \arrow[ul, "iQ_-"'] \arrow[u, dotted, "Q_A"] \arrow[ur, "\overline{Q}_+"] \arrow[r, dotted, "Q_B"] \arrow[dr, "i\overline{Q}_-"] \arrow[dl, "Q_+"'] \arrow[l, dotted, "\overline{Q}_B"] \arrow[d, dotted, "\overline{Q}_A"] & k_V \\
\mathcal{V}_{k_V-1,k_A-1} & {} & \mathcal{V}_{k_V+1,k_A-1}
\end{tikzcd} \tag{8.15b}$$

up to an overall factor $-1/2^{\frac{1}{2}}$.

The H twisted Laplacian is defined by

$$\Delta_H = [d_H, {d_H}^*]. \tag{8.16}$$

From (8.10), (6.25c), it follows readily that

$$\Delta_H = 2H_{QM}. \tag{8.17}$$

Thus, the twisted Laplacian equals twice the quantum Hamiltonian.

On view of (8.9), (8.10), (8.17), we see that the space of supersymmetric ground states of biHermitian supersymmetric quantum mechanics graded by the

[3] The above definitions of Q_A, Q_B differ from the conventional ones $Q_A = \overline{Q}_+ + Q_-$, $Q_B = \overline{Q}_+ + \overline{Q}_-$ by the factor i multiplying the second term. This factor is due to the phase choice conventions of the supercharges $Q_\pm$, $\overline{Q}_\pm$ we adopted. Note that the algebra (6.1) is invariant under the phase redefinitions $Q_\pm \to \mathrm{e}^{i\phi_\pm} Q_\pm$, $\overline{Q}_\pm \to \mathrm{e}^{-i\phi_\pm} \overline{Q}_\pm$. See also section 9.

values of the vector and axial R charges F_V, F_A can be identified with the complex d_H cohomology $H_H{}^\bullet(M,\mathbb{C})$, or equivalently, with the space of complex Δ_H harmonic differential forms $\mathrm{Harm}_H{}^\bullet(M,\mathbb{C})$ graded by $\mathcal{J}_1\cdot$, $\mathcal{J}_2\cdot$ eigenvalues. Such gradation constitutes the Hodge decomposition of the underlying twisted generalized Kaehler manifold M as defined in [33]:

$$H_H{}^\bullet(M,\mathbb{C}) = \mathrm{Harm}_H{}^\bullet(M,\mathbb{C}) = \bigoplus_{|k_1+k_2|\le n/2,\ k_1+k_2=n/2 \bmod 2} \mathcal{V}^{(0)}{}_{k_1,k_2}, \tag{8.18}$$

where $\mathcal{V}^{(0)}{}_{k_1,k_2}$ are Δ_H-harmonic forms in $\mathcal{V}_{k_1,k_2}$.

In the usual Kaehler case, which occurs when $K_+ = K_- = K$ and $H = 0$, the decomposition (8.18) differs from the familiar Dolbeault decomposition. In [38], Michelsohn called it the Clifford decomposition and showed that there is an orthogonal transformation u, called the Hodge automorphism, taking it to the usual Dolbeault decomposition. u is given by

$$u = \exp\Big(-\frac{i\pi}{4}h\Big)\exp\Big(\frac{\pi}{4}(l^* - l)\Big), \tag{8.19}$$

where h, l, l^* are the generators of the Kaehler $\mathfrak{sl}(2,\mathbb{C})$ symmetry algebra [4]. Indeed, it is straightforward to verify that u maps $\Omega^{p,q}(M)$ into $\mathcal{V}_{k_1,k_2}$ with

$$k_1 = -p + q, \tag{8.20a}$$

$$k_2 = +p + q - \frac{n}{2}. \tag{8.20b}$$

In the Kaehler case, combining (8.11), (6.5), one finds

$$\delta_+ = \frac{1}{2}(\partial - \overline{\partial}^*), \tag{8.21a}$$

$$\overline{\delta}_+ = \frac{1}{2}(\overline{\partial} - \partial^*), \tag{8.21b}$$

$$\delta_- = \frac{1}{2}(\partial + \overline{\partial}^*), \tag{8.21c}$$

$$\overline{\delta}_- = \frac{1}{2}(\overline{\partial} + \partial^*), \tag{8.21d}$$

[4] h, l, l^* are given explicitly by $h = n/2 - \epsilon^a\iota_a$, $l = -(1/2)K^{ab}(x)\iota_a\iota_b$ $l^* = (1/2)K_{ab}(x)\epsilon^a\epsilon^b$ and satisfy the commutation relations $[h,l] = 2l$, $[h,l^*] = -2l^*$, $[l,l^*] = h$.

where ∂, $\overline{\partial}$ are the Dolbeault differentials. From (8.12), one has then

$$\overline{\partial}_1 = \overline{\partial}, \tag{8.22a}$$

$$\overline{\partial}_2 = \frac{1}{2}(d + id^{c*}), \tag{8.22b}$$

where $d = \overline{\partial} + \partial$ is the de Rham differential and $d^c = i(\overline{\partial} - \partial)$. Thus, $\overline{\partial}_1$ is nothing but the customary Dolbeault differential $\overline{\partial}$, as expected. Conversely, $\overline{\partial}_2$ has no immediate simple interpretation, but, in fact, it is straightforwardly related to the de Rham differential d by the Hodge automorphism u defined in (8.19):

$$\overline{\partial}_2 = 2^{-\frac{1}{2}} \mathrm{e}^{-\frac{i\pi}{4}} u d u^{-1}. \tag{8.23}$$

To conclude, we remark that the Hodge theory worked out in [33] is slightly more general than the one reviewed above, as it allows also for twisting by a field $B \in \Omega^2(M)$. B should not be confused with the field b trivializing H in (3.3). B acts on $\mathcal{V}$ via the Clifford action

$$B\cdot = \frac{1}{2} B_{ab}(x) \epsilon^a \epsilon^b. \tag{8.24}$$

B twisting amounts to a similarity transformation

$$X_B = \exp(B\cdot) X \exp(-B\cdot) \tag{8.25}$$

of the relevant operators X on $\mathcal{V}$. In particular, twisting transforms d_H into $(d_H)_B = d_{H+dB}$. From (8.25), it is clear that B twisting preserves the untwisted formal algebraic structure leaving the above conclusions unchanged. However, the adjunction properties of the untwisted operators such as (6.12), (6.11), (6.13) are not enjoyed by their twisted versions, if one uses the Hilbert product defined in (7.5), but they do, if one also twists the product as follows

$$\langle \alpha, \beta \rangle_B = \langle \exp(-B\cdot)\alpha, \exp(-B\cdot)\beta \rangle / \langle \exp(-B\cdot)1, \exp(-B\cdot)1 \rangle, \tag{8.26}$$

with $\alpha, \beta \in \mathcal{V}$. $\langle , \rangle_B$ is called the Born–Infeld product in [33], since

$$\langle \exp(-B\cdot)1, \exp(-B\cdot)1 \rangle = \frac{1}{\mathrm{vol}(M)} \int_M d^n t \, g^{-\frac{1}{2}} \det(g + B). \tag{8.27}$$

has the characteristic Born–Infeld form.

9 The biHermitian A and B sigma models

The topological twisting of the biHermitian $(2,2)$ supersymmetric sigma model is achieved by shifting the spin of fermions either by $F_V/2$ or $F_A/2$, where F_V, F_A are the fermion's vector and axial R charges, respectively. The resulting topological sigma models will be called biHermitian A and B models, respectively. As is well known, at the quantum level, the R symmetries of the sigma model are spoiled by anomalies in general. The twisting can be performed only if the corresponding R symmetry is non anomalous. This happens provided the following conditions are satisfied [9]:

$$c_1(T_+^{1,0}M) - c_1(T_-^{1,0}M) = 0, \qquad \text{vector } R \text{ symmetry}, \tag{9.1a}$$

$$c_1(T_+^{1,0}M) + c_1(T_-^{1,0}M) = 0, \qquad \text{axial } R \text{ symmetry}. \tag{9.1b}$$

R symmetry anomaly cancellation, however, is not sufficient by itself to ensure the consistency of the twisting. Requiring the absence of BRST anomalies and the existence of a one–to–one state–operator correspondence implies further conditions discussed in [9] in the framework of generalized Calabi–Yau geometry.

To generate topological sigma models using twisting, we switch to the Euclidean version of the $(2,2)$ supersymmetric sigma model. Henceforth, Σ is a compact Riemann surface of genus ℓ_Σ. Further, the following formal substitutions are to be implemented

$$\partial_{++} \to \partial_z \tag{9.2a}$$

$$\partial_{--} \to \overline{\partial}_{\overline{z}} \tag{9.2b}$$

$$\chi_+{}^a \to \chi_{+\theta}{}^a \in C^\infty(\Sigma, \kappa_\Sigma{}^{\frac{1}{2}} \otimes x^* T_+^{1,0}M) \tag{9.2c}$$

$$\overline{\chi}_+{}^a \to \overline{\chi}_{+\theta}{}^a \in C^\infty(\Sigma, \kappa_\Sigma{}^{\frac{1}{2}} \otimes x^* T_+^{0,1}M) \tag{9.2d}$$

$$\chi_-{}^a \to \chi_{-\overline{\theta}}{}^a \in C^\infty(\Sigma, \overline{\kappa}_\Sigma{}^{\frac{1}{2}} \otimes x^* T_-^{1,0}M) \tag{9.2e}$$

$$\overline{\chi}_-{}^a \to \overline{\chi}_{-\overline{\theta}}{}^a \in C^\infty(\Sigma, \overline{\kappa}_\Sigma{}^{\frac{1}{2}} \otimes x^* T_-^{0,1}M) \tag{9.2f}$$

where $\kappa_\Sigma{}^{\frac{1}{2}}$ is any chosen spin structure (a square root of the canonical line bundle

κ_Σ of Σ).

The field content of the biHermitian A sigma model is obtained from that of the $(2,2)$ supersymmetric sigma model via the substitutions

$$\chi_{+\theta}{}^a \rightarrow \psi_{+z}{}^a \in C^\infty(\Sigma, \kappa_\Sigma \otimes x^* T_+^{1,0} M), \tag{9.3a}$$

$$\overline{\chi}_{+\theta}{}^a \rightarrow \overline{\chi}_{+}{}^a \in C^\infty(\Sigma, x^* T_+^{0,1} M), \tag{9.3b}$$

$$\chi_{-\overline{\theta}}{}^a \rightarrow \chi_{-}{}^a \in C^\infty(\Sigma, x^* T_-^{1,0} M). \tag{9.3c}$$

$$\overline{\chi}_{-\overline{\theta}}{}^a \rightarrow \overline{\psi}_{-\overline{z}}{}^a \in C^\infty(\Sigma, \overline{\kappa}_\Sigma \otimes x^* T_-^{0,1} M), \tag{9.3d}$$

The BRST symmetry variations of the A sigma model fields are obtained from those of the $(2,2)$ supersymmetric sigma model fields (cf. eq. (3.10)), by setting

$$\alpha^+ = \tilde{\alpha}^- = 0, \tag{9.4a}$$

$$\tilde{\alpha}^+ = \alpha^- = \alpha. \tag{9.4b}$$

Similarly, the field content of the biHermitian B sigma model is obtained from that of the $(2,2)$ supersymmetric sigma model via the substitutions

$$\chi_{+\theta}{}^a \rightarrow \psi_{+z}{}^a \in C^\infty(\Sigma, \kappa_\Sigma \otimes x^* T_+^{1,0} M), \tag{9.5a}$$

$$\overline{\chi}_{+\theta}{}^a \rightarrow \overline{\chi}_{+}{}^a \in C^\infty(\Sigma, x^* T_+^{0,1} M), \tag{9.5b}$$

$$\chi_{-\overline{\theta}}{}^a \rightarrow \psi_{-\overline{z}}{}^a \in C^\infty(\Sigma, \overline{\kappa}_\Sigma \otimes x^* T_-^{1,0} M), \tag{9.5c}$$

$$\overline{\chi}_{-\overline{\theta}}{}^a \rightarrow \overline{\chi}_{-}{}^a \in C^\infty(\Sigma, x^* T_-^{0,1} M). \tag{9.5d}$$

The BRST symmetry variations of the B sigma model fields are obtained from those of the $(2,2)$ supersymmetric sigma model fields, by setting

$$\alpha^+ = \alpha^- = 0, \tag{9.6a}$$

$$\tilde{\alpha}^+ = \tilde{\alpha}^- = \alpha. \tag{9.6b}$$

Inspection of the A, B twist prescriptions reveals that

$$A \text{ twist } \leftrightarrows \ B \text{ twist } \ \text{under} \ \ K_-{}^a{}_b \leftrightarrows -K_-{}^a{}_b. \tag{9.7}$$

The target space geometrical data $(g, H, K_\pm)$, $(g, H, \pm K_\pm)$ have precisely the same properties: they are both biHermitian structures. So, at the classical level, any statement concerning the A (B) model translates automatically into one concerning the B (A) model upon reversing the sign of K_- [5]. For this reason, below, we shall consider only the B twist, unless otherwise stated.

The twisted action S_t is obtained from the $(2,2)$ supersymmetric sigma model action S (3.1) implementing the substitutions (9.5). One finds [21]

$$S_t = \int_\Sigma d^2z \bigg[\frac{1}{2}(g_{ab}+b_{ab})(x)\partial_z x^a \overline{\partial}_{\overline{z}} x^b \tag{9.8}$$

$$+ ig_{ab}(x)(\psi_{+z}{}^a \overline{\nabla}_{+\overline{z}} \overline{\chi}_+{}^b + \psi_{-\overline{z}}{}^a \nabla_{-z} \overline{\chi}_-{}^b)$$

$$+ R_{+abcd}(x)\overline{\chi}_+{}^a \psi_{+z}{}^b \overline{\chi}_-{}^c \psi_{-\overline{z}}{}^d\bigg].$$

The topological field variations are obtained from the $(2,2)$ supersymmetry field variations (3.10) via (9.5), (9.6). In (9.6b), there is no real need for the supersymmetry parameters $\tilde{\alpha}^+$, $\tilde{\alpha}^-$ to take the same value α, since, under twisting both become scalars. If we insist $\tilde{\alpha}^+$, $\tilde{\alpha}^-$ to be independent in (3.4), we obtain a more general symmetry variation

$$\delta_t = \tilde{\alpha}^+ s_{t+} + \tilde{\alpha}^- s_{t+} \tag{9.9}$$

where the fermionic variation operators $s_{t\pm}$ are given by [21]

$$s_{t+} x^a = i\overline{\chi}_+{}^a, \tag{9.10a}$$

$$s_{t-} x^a = i\overline{\chi}_-{}^a, \tag{9.10b}$$

$$s_{t+} \overline{\chi}_+{}^a = 0, \tag{9.10c}$$

$$s_{t-} \overline{\chi}_+{}^a = -i\Gamma_+{}^a{}_{bc}(x)\overline{\chi}_-{}^b \overline{\chi}_+{}^c, \tag{9.10d}$$

$$s_{t+} \overline{\chi}_-{}^a = -i\Gamma_-{}^a{}_{cb}(x)\overline{\chi}_+{}^c \overline{\chi}_-{}^b, \tag{9.10e}$$

$$s_{t-} \overline{\chi}_-{}^a = 0, \tag{9.10f}$$

[5] For notational consistency, exchanging $K_-{}^a{}_b \leftrightarrows -K_-{}^a{}_b$ must be accompanied by switching $\alpha^- \leftrightarrows \tilde{\alpha}^-$.

$$s_{t+}\psi_{+z}{}^a = -i\Gamma_+{}^a{}_{bc}(x)\overline{\chi}_+{}^b\psi_{+z}{}^c - \Lambda_+{}^a{}_b(x)(\partial_z x^b - iH^b{}_{cd}(x)\overline{\chi}_+{}^c\psi_{+z}{}^d), \qquad (9.10g)$$

$$s_{t-}\psi_{+z}{}^a = -i\Gamma_+{}^a{}_{bc}(x)\overline{\chi}_-{}^b\psi_{+z}{}^c, \qquad (9.10h)$$

$$s_{t+}\psi_{-\overline{z}}{}^a = -i\Gamma_-{}^a{}_{bc}(x)\overline{\chi}_+{}^b\psi_{-\overline{z}}{}^c, \qquad (9.10i)$$

$$s_{t-}\psi_{-\overline{z}}{}^a = -i\Gamma_-{}^a{}_{bc}(x)\overline{\chi}_-{}^b\psi_{-\overline{z}}{}^c - \Lambda_-{}^a{}_b(x)(\overline{\partial}_{\overline{z}} x^b + iH^b{}_{cd}(x)\overline{\chi}_-{}^c\psi_{-\overline{z}}{}^d). \qquad (9.10j)$$

The action S_t is invariant under both $s_{t\pm}$ [21],

$$s_{t\pm}S_t = 0. \qquad (9.11)$$

One can show also that the $s_{t\pm}$ are nilpotent and anticommute on shell

$$s_{t\pm}{}^2 \approx 0, \qquad (9.12a)$$

$$s_{t+}s_{t-} + s_{t-}s_{t+} \approx 0, \qquad (9.12b)$$

where $\approx$ denotes equality on shell. The remarks following eqs. (4.13) hold in this case as well. The usual topological BRST variation s_t is obtained when (9.6b) is satisfied. s_t and the $s_{t\pm}$ are related as

$$s_t = s_{t+} + s_{t-}. \qquad (9.13)$$

(9.13) corresponds to the decomposition of the BRST charge in its left and right chiral components. Clearly, one has

$$s_t{}^2 \approx 0. \qquad (9.14)$$

Therefore, the $s_{t\pm}$ define an on shell cohomological double complex, whose total differential is s_t, a fact already noticed in [9]. The total on shell cohomology is isomorphic to the BRST cohomology.

Taking (9.5) into account and comparing (9.10) and (4.4), we see that, in the limit in which the world sheet Σ shrinks to a world line T, yielding the point particle approximation leading to biHermitian supersymmetric quantum mechanics, the sigma model variation operators $s_{t\pm}$ correspond to the quantum mechanics variation operators $\overline{q}_\pm$ and, so, in the full quantum theory, to the supercharges $\overline{Q}_\pm$. Thus, the $s_{t\pm}$ correspond to the operators $\overline{\delta}_\pm$ of Gualtieri's generalized Hodge theory via (8.11). This solves the problem of interpreting the $s_{t\pm}$ in the framework of generalized Kaehler geometry, which was posed but not solved in [21].

By (9.13), the topological BRST variation s_t is the counterpart of the supercharge $\overline{Q}_+ + \overline{Q}_-$. This is almost the supercharge $Q_B = \overline{Q}_+ + i\overline{Q}_-$ defined in (8.14a). To turn it precisely in this form, we perform the phase redefinitions $\overline{\chi}_-{}^a \to +i\overline{\chi}_-{}^a$, $\psi_{-\overline{z}}{}^a \to -i\psi_{-\overline{z}}{}^a$ and $s_{t-} \to +is_{t-}$. These redefintions leave both the action S_t, eq. (9.8), and the variation operators $s_{t\pm}$, eqs. (9.10), invariant in form, as is easy to see, but lead to identifying s_{t-} with $i\overline{Q}_-$ rather than $\overline{Q}_-$. Upon doing this, s_t corresponds to the supercharge Q_B and thus to the Lie algebroid differential $\overline{\partial}_1$ of Gualtieri's theory via (8.13). Thus, we recover one of the main results of reference [9].

Our analysis so far concerned the state BRST cohomology. One may also consider the operator BRST cohomology. In a topological field theory, the state and operator BRST complexes are isomorphic and, so, are their BRST cohomologies. In [9], it was shown that, in order such correspondence to hold, the topological condition (9.1b) is not sufficient (for the B model). It is necessary to require that $\mathcal{J}_1$ is a weak twisted generalized Calabi–Yau structure. Let us recall briefly the meaning of this notion.

Let E_1 be the $-i$ eigenbundle of the twisted generalized complex structure $\mathcal{J}_1$ in $(TM \oplus T^*M) \otimes \mathbb{C}$. With E_1, there is associated locally a nowhere vanishing section ρ_1 of $\wedge^*T^*M \otimes \mathbb{C}$ defined up to pointwise normalization by the condition $s \cdot \rho_1 = 0$, for all sections s of E_1, where $\cdot$ denotes the Clifford action (8.1). Globally, ρ_1 defines a line bundle U_1 in $\wedge^*T^*M\otimes\mathbb{C}$, called the canonical line bundle of $\mathcal{J}_1$. By definition, $\mathcal{J}_1$ is a weak twisted generalized Calabi–Yau structure, if U_1 is twisted generalized holomorphically trivial. This means that: *a*) U_1 is topologically trivial and, so, admits a global nowhere vanishing section ρ_1, which is a form in $\Omega^*(M) \otimes \mathbb{C}$; *b*) ρ_1 is twisted generalized holomorphic,

$$\overline{\partial}_1 \rho_1 = 0. \tag{9.15}$$

By part a, there exists a linear isomorphism $\phi : C^\infty(M, \wedge^* \overline{E}_1) \to \Omega^*(M) \otimes \mathbb{C}$ defined by the relation

$$\phi(s) = s \cdot \rho_1, \tag{9.16}$$

with s a section of $\wedge^* \overline{E}_1$. By part b, ϕ has the property that

$$\overline{\partial}_1 \phi(s) = \phi(\partial_{\overline{E}_1} s), \tag{9.17}$$

where $\partial_{\overline{E}_1}$ is the Lie algebroid differential of the Lie algebroid $\overline{E}_1$. Therefore, ϕ yields an isomorphism of the differential complexes $(C^\infty(M, \wedge^* \overline{E}_1), \partial_{\overline{E}_1})$ and $(\Omega^*(M) \otimes \mathbb{C}, \overline{\partial}_1)$ and, so, of their cohomologies. The canonical line bundle U_1 is isomorphic to the determinant line bundle $\det E_1$, $U_1 \simeq \det E_1$. The condition (9.1b) is equivalent to $c_1(E_1) = 0$ and, thus, to the triviality of $\det E_1$. Therefore, (9.1b) is equivalent only to part a of the weak twisted generalized Calabi–Yau condition and, so, it implies the existence of an isomorphism ϕ satisfying (9.16), but not (9.17). Part b has the further consequence (9.17), that leads to aforementioned cohomology isomorphism.

As shown in [9] and reviewed above, $(\Omega^*(M) \otimes \mathbb{C}, \overline{\partial}_1)$ is just the state BRST complex $(\mathcal{V}, Q_B)$. By the basic relation $(\partial_{\overline{E}_1} s)\cdot = [\overline{\partial}_1, s\cdot]$, $(C^\infty(M, \wedge^* \overline{E}_1), \partial_{\overline{E}_1})$ may be identified with the operator BRST complex $(\mathcal{O}, \widehat{Q}_B)$, where $\mathcal{O}$ is the sigma model operator algebra and

$$\widehat{Q}_B O = [Q_B, O], \tag{9.18}$$

with O any operator. This identification hinges on the actual content of the operator algebra $\mathcal{O}$, which we have not specified. Alternatively, one may use it as a definition of $\mathcal{O}$.

Now, we know that we also have the complexes $(\Omega^*(M) \otimes \mathbb{C}, \overline{\delta}_\pm)$ or, physically, the state complexes $(\mathcal{V}, \overline{Q}_\pm)$. To the $\overline{\delta}_\pm$, there should correspond nilpotent differentials $\delta_{\overline{E}_1\pm}$ in $C^\infty(M, \wedge^* \overline{E}_1)$ defined by

$$\overline{\delta}_\pm \phi(s) = \phi(\delta_{\overline{E}_1\pm} s), \tag{9.19}$$

with s a section of $\wedge^* \overline{E}_1$. Note that $(\delta_{\overline{E}_1\pm} s)\cdot = [\overline{\delta}_\pm, s\cdot]$. It is not obvious *a priori* that this really works, since the commutator in the right hand side of this relation is in principle a first order differential operator, but it actually does, as is easy to verify directly using the explicit expressions of $\overline{\delta}_\pm$ obtainable from (8.11), (6.5b). In this way, we have obtained differential complexes $(C^\infty(M, \wedge^* \overline{E}_1), \delta_{\overline{E}_1\pm})$. On the physical side, this should correspond to operator differential complexes $(\mathcal{O}, \widehat{\overline{Q}}_\pm)$, where $\mathcal{O}$ is as above and the $\widehat{\overline{Q}}_\pm$ are defined by

$$\widehat{\overline{Q}}_\pm O = [\overline{Q}_\pm, O], \tag{9.20}$$

with O any operator.

The above considerations indicate that the variation operators $s_{t\pm}$ do not simply characterize the topological sigma model at the classical level, but have an operator counterpart $\overline{Q}_\pm$ at the quantum level. The cohomologies of Q_B, $\overline{Q}_\pm$ are pairwise isomorphic. Ultimately, the topological correlators are expected to depend only on their common cohomology.

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A Formulae of biHermitian geometry

In this appendix, we collect a number of useful identities of biHermitian geometry, which are repeatedly used in the calculations illustrated in the main body of the paper. Below $(g, H, K_\pm)$ is a fixed biHermitian structure on an even dimensional manifold M.

1. Relations satisfied by the 3–form H_{abc}.

$$\partial_a H_{bcd} - \partial_b H_{acd} + \partial_c H_{abd} - \partial_d H_{abc} = 0. \tag{A.1}$$

2. Relations satisfied by the connections $\Gamma_\pm{}^a{}_{bc}$.

$$\Gamma_\pm{}^a{}_{bc} = \Gamma^a{}_{bc} \pm \frac{1}{2} H^a{}_{bc}, \tag{A.2a}$$

$$\Gamma_\pm{}^a{}_{bc} = \Gamma_\mp{}^a{}_{cb}, \tag{A.2b}$$

   where $\Gamma^a{}_{bc}$ is the Levi–Civita connection of the metric g_{ab}.

3. Relations satisfied by the torsion $T_\pm{}^a{}_{bc}$ of $\Gamma_\pm{}^a{}_{bc}$.

$$T_\pm{}^a{}_{bc} = \pm H^a{}_{bc}, \tag{A.3a}$$

$$T_\pm{}^a{}_{bc} = T_\mp{}^a{}_{cb}. \tag{A.3b}$$

4. Relations satisfied by the Riemann tensor $R_{\pm abcd}$ of $\Gamma_\pm{}^a{}_{bc}$.

$$R_{\pm abcd} = R_{abcd} \pm \frac{1}{2}(\nabla_d H_{abc} - \nabla_c H_{abd}) + \frac{1}{4}(H^e{}_{ad} H_{ebc} - H^e{}_{ac} H_{ebd}), \tag{A.4a}$$

$$R_{\pm abcd} = R_{\mp cdab}, \tag{A.4b}$$

   where R_{abcd} is the Riemann tensor of the metric g_{ab}.

   Bianchi identities.

$$\begin{aligned} R_{\pm abcd} + R_{\pm acdb} + R_{\pm adbc} \mp (\nabla_{\pm b} H_{acd} + \nabla_{\pm c} H_{adb} + \nabla_{\pm d} H_{abc}) \\ + H^e{}_{ab} H_{ecd} + H^e{}_{ac} H_{edb} + H^e{}_{ad} H_{ebc} = 0, \end{aligned} \tag{A.5a}$$

$$\begin{aligned} \nabla_{\pm e} R_{\pm abcd} + \nabla_{\pm c} R_{\pm abde} + \nabla_{\pm d} R_{\pm abec} \\ \pm (H^f{}_{ec} R_{\pm abfd} + H^f{}_{cd} R_{\pm abfe} + H^f{}_{de} R_{\pm abfc}) = 0. \end{aligned} \tag{A.5b}$$

Other identities

$$R_{\pm abcd} - R_{\pm cbad} = R_{\pm acbd} \pm \nabla_{\pm d} H_{abc}, \tag{A.6a}$$

$$R_{\pm abcd} - R_{\pm cbad} = R_{\mp acbd} \mp \nabla_{\mp b} H_{acd}, \tag{A.6b}$$

$$\begin{aligned} R_{\pm abcd} - R_{\mp abcd} = \pm \nabla_{\pm d} H_{abc} \mp \nabla_{\pm c} H_{dab} \\ + H^e{}_{ac} H_{ebd} + H^e{}_{da} H_{ebc} - H^e{}_{ab} H_{ecd}. \end{aligned} \tag{A.6c}$$

5. The complex structures $K_\pm{}^a{}_c K_\pm{}^c{}_b$.

$$K_\pm{}^a{}_c K_\pm{}^c{}_b = -\delta^a{}_b. \tag{A.7}$$

Integrability

$$K_\pm{}^d{}_a \partial_d K_\pm{}^c{}_b - K_\pm{}^d{}_b \partial_d K_\pm{}^c{}_a - K_\pm{}^c{}_d \partial_a K_\pm{}^d{}_b + K_\pm{}^c{}_d \partial_b K_\pm{}^d{}_a = 0. \tag{A.8}$$

Hermiticity

$$g_{cd} K_\pm{}^c{}_a K_\pm{}^d{}_b = g_{ab}. \tag{A.9}$$

Kaehlerness with torsion

$$\nabla_{\pm a} K_\pm{}^b{}_c = 0. \tag{A.10}$$

6. Other properties.

$$H_{efg} \Lambda_\pm{}^e{}_a \Lambda_\pm{}^f{}_b \Lambda_\pm{}^g{}_c = 0 \text{ and c. c.}, \tag{A.11}$$

$$R_{\pm efcd} \Lambda_\pm{}^e{}_a \Lambda_\pm{}^f{}_b = 0 \text{ and c. c.}, \tag{A.12}$$

where

$$\Lambda_\pm{}^a{}_b = \frac{1}{2}(\delta^a{}_b - i K_\pm{}^a{}_b) \text{ and c. c.} \tag{A.13}$$

B The spectrum of F_V, F_A

Define the operators

$$F_\pm = g_{ab}(x)\overline{\chi}_\pm{}^a \chi_\pm{}^b, \tag{B.1a}$$

$$\tilde{F}_\pm = g_{ab}(x)\chi_\pm{}^a \overline{\chi}_\pm{}^b. \tag{B.1b}$$

These operators are selfadjoint and non negative

$$F_\pm = F_\pm{}^* \geq 0, \tag{B.2a}$$

$$\tilde{F}_\pm = \tilde{F}_\pm{}^* \geq 0. \tag{B.2b}$$

Further they satisfy the relation

$$F_\pm + \tilde{F}_\pm = \frac{n}{2} \tag{B.3}$$

From (6.4), it is easy to see that

$$F_V = F_+ + F_- - \frac{n}{2} = -\tilde{F}_+ - \tilde{F}_- + \frac{n}{2}, \tag{B.4a}$$

$$F_A = F_+ - F_- = -\tilde{F}_+ + \tilde{F}_-. \tag{B.4b}$$

From (B.2), (B.4), it is immediate to see that the joint F_V, F_A eigenspaces $\mathcal{V}_{k_V,k_A}$ vanish unless the two conditions (6.16a) hold.

For fixed k_V, the spaces $\mathcal{V}_{k_V,k_A}$ are obtained from $\mathcal{V}_{k_V,-n/2+|k_V|}$ by applying operators with vanishing F_V charge and positive F_A charge. The only such operators are linear combinations $\chi_-{}^a\overline{\chi}_+{}^b$ with F_V, F_A neutral coefficients or products of such operators. These operators increase the eigenvalue of F_A by multiples of 2. A similar reasoning holds interchanging the roles of k_V, k_A.

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